ON THE JACOBIAN ELLIPTIC FUNCTIONS.

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1. A GENERALIZED TYPE OF CYCLIC FUNCTIONS.

In a work on Algebra* recently published, I have defined a generalized type of sine and cosine in which a complex base takes the place of e, or natural base. I define them with reference to a modulus, an apparition that subsequently reappears as the modulus of a modified form of the Jacobian elliptic functions, which these generalized cyclic functions serve to define.

With respect to the modulus $\varkappa$, itself defined by the identity

$$\varkappa \equiv |\varkappa| (\cos\beta + i\sin\beta)\,,$$

the definitions of sine and cosine are

$$\sin_\kappa w \equiv \frac{\varkappa}{2}(e^{w/\kappa} - e^{-w/\kappa})\,,$$

$$\cos_\kappa w \equiv \frac{1}{2}(e^{w/\kappa} + e^{-w/\kappa})\,,$$

in which w is in general complex. The functions $\tan_\kappa w$, $\cot_\kappa w$, $\sec_\kappa w$, $\csc_\kappa w$, are defined, in the usual way, in terms of $\sin_\kappa w$ and $\cos_\kappa w$. These are *modulo-cyclic*, or more briefly, *modo-cyclic* functions.

The analytical theory of these functions is quite as simple as that of either the circular, or the hyperbolic forms of them, which they in fact become, without further reduction, by the substitutions $\varkappa = i = \sqrt{-1}$ and $\varkappa = 1$ respectively; and we may find it advantageous to substitute for the double circular-hyperbolic theory, with its double series of formulæ, the unique modo-cyclic theory and its inclusive table of properties, deriving the former from the latter by the above-mentioned rule of derivation.† For example, it is simpler to remember this rule and write

$$\cos^2_\kappa w - \varkappa^{-2}\sin^2_\kappa w = 1\,,$$

$$\sin_\kappa(w \pm w') = \sin_\kappa w \cos_\kappa w' \pm \cos_\kappa w \cos_\kappa w'\,,$$

$$\cos_\kappa(w \pm w') = \cos_\kappa w \cos_\kappa w' \pm \varkappa^{-2}\sin_\kappa w \sin_\kappa w'\,,$$

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* *Uniplanar Algebra*, p. 101.

† *Uniplanar Algebra*, pp 102–106.

than to write out the six corresponding circular and hyperbolic formulæ, remembering the variations in algebraic sign for this purpose. The following are the relations, obvious corollaries from the definitions, that connect the various functions with each other:

$$\sin_\kappa w = \kappa \sinh \frac{w}{\kappa}, \qquad \cos_\kappa w = \cosh \frac{w}{\kappa},$$

$$\sin_\kappa w = i\kappa \sin \frac{w}{i\kappa}, \qquad \cos_\kappa w = \cos \frac{w}{i\kappa},$$

$$\sin_1 w = \sinh w, \qquad \cos_1 w = \cosh w,$$

$$\sin_i w = \sin w, \qquad \cos_i w = \cos w.$$

The derivatives of $\sin_\kappa$ and $\cos_\kappa$ are

$$\frac{d \sin_\kappa w}{dw} = \cos_\kappa w, \qquad \frac{d \cos_\kappa w}{dw} = \kappa^{-2} \sin_\kappa w.$$

So much is sufficient to properly characterize these modo-cyclic functions and to indicate their place in the list of the ordinary transcendental functions. The more important question of how they behave when made the vehicles of higher analytical processes finds its answer in part in the following brief investigation, which deals primarily with the definitions of the elliptic functions.

2. Rationalization of $dv/\sqrt{a + 2bv + cv^2}$.

As preliminary to the more general discussion, the rationalization of the simpler differential expression $dv/\sqrt{a + 2bv + cv^2}$, by means of the $\sin_\kappa$-function, is suggestive.

Let

$$V = a + 2bv + cv^2,$$

and suppose this reduced to the form

$$Z = a'(1 + \kappa^{-2} z^2),$$

by the substitution

$$v = \lambda + \mu z.$$

For this purpose it is sufficient to make $\lambda = - b/c$, whence, by a comparison of coefficients,

$$a' = (ac - b^2)/c,$$

$$\kappa^{-2} = c^2\mu^2/(ac - b^2),$$

in which μ is still arbitrary, so that by properly choosing μ, κ may be made to assume any desired value.

The differential expression then becomes

$$\frac{\mu}{\sqrt{a'}} \cdot \frac{dz}{\sqrt{1 + \varkappa^{-2} z^2}},$$

and

$$\frac{dv}{\sqrt{V}} = \frac{1}{\varkappa\sqrt{c}} \cdot \frac{dz}{\sqrt{1 + \varkappa^{-2} z^2}}.$$

The final substitution may now be made in the form

$$z = \sin_\kappa \left[\varphi + ri\varkappa\frac{\pi}{2}\right];$$

in which r is eventually to be chosen either $= 0$, or $= 1$. The rationalization takes place by virtue of the relation

$$1 + \varkappa^{-2} \sin_\kappa{}^2 w = \cos_\kappa{}^2 w\,;$$

and since

$$dz = \cos_\kappa \left[\varphi + ri\varkappa\frac{\pi}{2}\right] d\varphi,$$

and

$$z = \frac{\varkappa(cv + b)}{\sqrt{ac - b^2}},$$

we have, at once,

$$\frac{dv}{\sqrt{V}} = \frac{d\varphi}{\varkappa\sqrt{c}},$$

or

$$\int\frac{dv}{\sqrt{V}} = \frac{1}{\varkappa\sqrt{c}}\left[\sin_\kappa{}^{-1}\frac{\varkappa(cv + b)}{\sqrt{ac - b^2}} - ri\varkappa\frac{\pi}{2}\right].$$

In the ordinary case, when the variables and coefficients are all real, there are four alternatives to be considered, and in each the value of μ is to be so chosen, if possible, as to make the result real.

(i). If c and $ac - b^2$ are both positive, take $r = 0$ and μ such that $\varkappa = 1$; then

$$\int\frac{dv}{\sqrt{V}} = \frac{1}{\sqrt{c}}\sinh^{-1}\frac{cv + b}{\sqrt{ac - b^2}}.$$

(ii). If c and $ac - b^2$ are both negative, take $r = 0$ and μ such that $\varkappa = i$; then

$$\int\frac{dv}{\sqrt{V}} = -\frac{1}{\sqrt{-c}}\sin^{-1}\frac{cv + b}{\sqrt{b^2 - ac}}.$$

(iii). If $c > 0$ and $ac - b^2 < 0$, take $r = 1$ and μ such that $\varkappa = 1$; then, since $\sin_\kappa\left[\varphi + i\varkappa\frac{\pi}{2}\right] = i\varkappa\cos_\kappa\varphi$,

$$\int\frac{dv}{\sqrt{V}} = \frac{1}{\sqrt{c}}\cosh^{-1}\frac{cv + b}{\sqrt{b^2 - ac}}.$$

(iv). If $c < 0$ and $ac - b^2 > 0$, take μ such that $\varkappa = 1$; then

$$\int \frac{dv}{\sqrt{V}} = \frac{1}{\sqrt{c}} \left[\sinh^{-1} \frac{cv + b}{\sqrt{ac - b^2}} - ri\frac{\pi}{2} \right].$$

This involves, as it should, an imaginary term, whether $r = 0$, or 1.

3. Reduction of $dv/\sqrt{V}$ to a Normal Form:

$$V \equiv a + 4bv + 6cv^2 + 4dv^3 + ev^4.$$

For the reduction of the elliptic differential $dv/\sqrt{V}$ to a normal form I employ Cayley's linear transformation-theory, first published in the *Cambridge and Dublin Mathematical Journal*, Vol. I (1846), pp. 70–73, reprinted in Cayley's *Collected Mathematical Papers*, Vol. I, pp. 224–227, and also in abridged form in his *Elliptic Functions*, pp. 317–320. Though the first part of the transformation here presented is identical with Cayley's, for completeness' sake so much of the latter is reproduced as is necessary to preserve continuity in the discussion.

Let $(x, y)^4$ denote a quartic in x, y, of the form

$$(x, y)^4 \equiv ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4,$$

and let this be transformed into

$$(x', y')^4 \equiv a'x'^4 + 4b'x'^3y' + 6c'x'^2y'^2 + 4d'x'y'^3 + e'y'^4,$$

by the linear substitution

$$x = \lambda_1 x' + \mu_1 y', \quad y = \lambda_2 x' + \mu_2 y'.$$

The differential expressions

$$xdy - ydx, \quad \frac{xdy - ydx}{\sqrt{(x, y)^4}}$$

are hereby transformed into

$$xdy - ydx = (\lambda_1\mu_2 - \lambda_2\mu_1)(x'dy' - y'dx')$$

and

$$\frac{xdy - ydx}{\sqrt{(x, y)^4}} = (\lambda_1\mu_2 - \lambda_2\mu_1)\frac{x'dy' - y'dx'}{\sqrt{(x', y')^4}}.$$

If x, y, x', y' be now replaced by

$$v = y/x, \quad v' = y'/x',$$

and if

$$V \equiv a + 4bv + 6cv^2 + 4dv^3 + ev^4,$$

$$V' \equiv a' + 4b'v' + 6c'v'^2 + 4d'v'^3 + e'v'^4,$$

$$m \equiv \lambda_1\mu_2 - \lambda_2\mu_1,$$

the differential equation becomes

$$\frac{dv}{\sqrt{V}} = m\frac{dv'}{\sqrt{V'}},$$

and the substitution for this transformation is

$$v = \frac{\lambda_2 + \mu_2 v'}{\lambda_1 + \mu_1 v'}.$$

Since $(x', y')^4$ has been derived from $(x, y)^4$ by the linear transformation

$$x = \lambda_1 x' + \mu_1 y', \quad y = \lambda_2 x' + \mu_2 y',$$

between the invariants

$$g_2 \equiv ae - 4bd + 3c^2,$$

$$g_3 \equiv ace + 2bcd - ad^2 - eb^2 - c^3,$$

and the corresponding invariants g'_2, g'_3, of $(x', y')^4$, there exist the well-known relations

$$g'_2 = m^4 g_2, \quad g'_3 = m^6 g_3, \quad g'^3_2/g'^2_3 = g_2^3/g_3^2.$$

This much is identical with Cayley's original transformation. In the immediate sequel Cayley's method is continued in the reduction of V' to the form

$$4b'v'(1 + pv')(1 + qv'),$$

instead of the form $a'(1 + pv'^2)(1 + qv'^2)$, and in the determination of the modulus belonging to it.

The reduced form of the differential we seek is one in which V' is of the third degree, and we assume it to be

$$V' \equiv 4b'v'(1 + pv')(1 + qv').$$

Comparing the coefficients of this form with those of

$$V' \equiv a' + 4b'v' + 6c'v'^2 + 4d'v'^3 + e'v'^4,$$

we find that

$$a' = 0, \qquad c' = \frac{2}{3}b'(p + q),$$

$$e' = 0, \qquad d' = b'pq;$$

whence

$$g'_2 = -4b'^2 pq + \frac{4}{3}b'^2(p + q)^2$$

$$= \frac{4}{3}b'^2(p^2 + q^2 - pq),$$

and

$$g'_3 = 2b' \cdot \frac{2}{3} b'(p+q) \cdot b'pq - \frac{8}{27} b'^3 (p+q)^3$$
$$= \frac{4}{27} b'^3 (p+q)(5pq - 2p^2 - 2q^2).$$

Hence the equation connecting p and q is

$$\frac{g'^2_3}{g'^3_2} = \frac{(p+q)^2(5pq-2p^2-2q^2)^2}{4 \cdot 27 \cdot (p^2+q^2-pq)^3} = \frac{g_3^2}{g_2^3},$$

involving, other than p and q, only the absolute invariant g_3^2/g_2^3 and numerical coefficients.

This equation will be somewhat simplified by subtracting its two members from unity, becoming thereby

$$1 - \frac{27 g_3^2}{g_2^3} = \frac{4(p^2+q^2-pq)^3 - (p+q)^2(5pq-2p^2-2q^2)^2}{4(p^2+q^2-pq)^3},$$

an equation which, for the purpose of further simplification, may be expressed wholly in terms of $p-q$ and pq, thus:

$$\frac{\Delta}{g_2^3} = \frac{4\{(p-q)^2+pq\}^3 - \{(p-q)^2+4pq\}\{pq-2(p-q)^2\}^2}{4\{(p-q)^2+pq\}^3},$$

where

$$\Delta = g_2^3 - 27 g_3^2;$$

or, if $p - q = \varphi$ and $pq = \psi$, this may be written

$$\frac{\Delta}{g_2^3} = \frac{4(\varphi^2+\psi)^3 - (\varphi^2+4\psi)(\psi-2\varphi^2)^2}{4(\varphi^2+\psi)^3}.$$

When the numerator of this fraction is expanded, all the terms except the three that involve $\varphi^2\psi^2$ mutually cancel one another and the result is

$$\frac{\Delta}{g_2^3} = \frac{27\,\varphi^2\psi^2}{4(\varphi^2+\psi)^3} = \frac{27\,p^2q^2(p-q)^2}{4(p^2+q^2-pq)^3}.$$

This is a reciprocal equation of the sixth degree and has three roots of the form q/p and three of the reciprocal form p/q. In order to its further simplification let

$$R = \frac{27}{4} \cdot \frac{g_2^3}{\Delta},$$

$$\sqrt{\theta - 1} = \sqrt{q/p} - \sqrt{p/q},$$

substitutions that lead at once to a cubic equation in θ from whose roots the values of q/p are easily obtained. In fact, the sextic equation in p, q now is

$$(p^2+q^2-pq)^3 - Rp^2q^2(p-q)^2 = 0,$$

and

$$(p-q)^2 = pq(\theta - 1),$$

$$p^2 + q^2 - pq = pq\theta;$$

$$\therefore \quad \theta^3 - R(\theta - 1) = 0.$$

This is Cayley's cubic. (See *Elliptic Functions*, p. 319.)

The values of q/p in terms of θ are found to be

$$\frac{q}{p} = \frac{1}{2}\{\theta + 1 \pm \sqrt{(\theta - 1)(\theta + 1)}\};$$

and the two values contained in this formula are reciprocal to each other, a statement easily verified by showing that their product is unity.

For the determination of $m/\sqrt{b'}$, which is a factor in the differential expression $mdv'/\sqrt{V'}$, in terms of p and q, we have the equation

$$g'_2 = \frac{4}{3}b'^2(p^2 + q^2 - pq) = m^4 g_2,$$

from which we derive

$$\frac{m}{\sqrt{b'}} = \left[\frac{4}{3}\cdot\frac{p^2 + q^2 - pq}{g_2}\right]^{\frac{1}{4}};$$

whence

$$\frac{dv}{\sqrt{V}} = \left[\frac{p^2 + q^2 - pq}{12g_2}\right]^{\frac{1}{4}} \frac{dv'}{\sqrt{v'(1 + pv')(1 + qv')}};$$

or, since

$$p^2 + q^2 - pq = pq\theta,$$

$$\frac{dv}{\sqrt{V}} = \left[\frac{pq\theta}{12g_2}\right]^{\frac{1}{4}} \frac{dv'}{\sqrt{v'(1 + pv')(1 + qv')}}.$$

The further substitutions

$$qv' = z, \quad q/p = \varkappa^2$$

lead to the following final normal form of the elliptic differential of the first kind:

$$\frac{dv}{\sqrt{V}} = \left[\frac{\theta\varkappa^{-2}}{12g_2}\right]^{\frac{1}{4}} \frac{dz}{\sqrt{z(z + 1)(\varkappa^{-2}z + 1)}},$$

whose modulus $\varkappa$ is determined by the equations

$$\varkappa = \frac{1}{2}(\sqrt{\theta - 1} \pm \sqrt{\theta + 3}),$$

$$\theta^3 - \frac{27g_2^3}{4\Delta}(\theta - 1) = 0.$$

It is easily shown that the six values of q/p, or x^2, obtained as the solution of the above sextic equation, are the six anharmonic ratios formed with the differences of the roots of the original quartic equation $V=0$.

4. The Co-efficients of Transformation.

In terms of v the new variable z is

$$z = qv' = q \cdot \frac{\lambda_2 - \lambda_1 v}{\mu_1 v - \mu_2},$$

and for the determination of q we have the equations

$$\frac{(\lambda_1\mu_2 - \lambda_2\mu_1)^2}{b'} = 4\left[\frac{pq\theta}{12g_2}\right]^{\frac{1}{2}} = 4q\left[\frac{\theta x^{-2}}{12g_2}\right]^{\frac{1}{2}},$$

$$(\lambda_1\mu_2 - \lambda_2\mu_1) = \frac{\lambda_1^2}{g}(\mu - \lambda),$$

where

$$\lambda = \lambda_2/\lambda_1, \qquad \mu = \mu_2/\mu_1, \qquad g = \lambda_1/\mu_1,$$

while b', as determined from the linear transformation of $(x, y)^4$ into $(x', y')^4$, by comparing coefficients, is

$$b' = \lambda_1^3\mu_1\{a + b(3\lambda + \mu) + 3c(\lambda + \mu)\lambda + d(\lambda + 3\mu)\lambda^2 + e\mu\lambda^3\}.$$

From these equations the value of gq is obtained in the form

$$gq = \frac{x\lambda_1^4}{4b'g}(\mu - \lambda)^2\sqrt{\frac{12g_2}{\theta}}$$

$$= \frac{x(\mu - \lambda)^2}{4B}\sqrt{\frac{12g_2}{\theta}},$$

in which

$$B \equiv a + b(3\lambda + \mu) + 3c(\lambda + \mu)\lambda + d(\lambda + 3\mu)\lambda^2 + e\mu\lambda^3.$$

Hence, the final form of z in terms of v and known constants is

$$z = \frac{x(\mu - \lambda)^2}{4B} \cdot \sqrt{\frac{12g_2}{\theta}} \cdot \frac{\lambda - v}{v - \mu},$$

λ and μ being the two roots of the quartic equation $V=0$, corresponding to the roots 0 and ∞, of $V'=0$, regarded as a quartic in v', a remark at once verified by observing that, in the formula of transformation

$$v = \frac{\lambda_2 + \mu_2 v'}{\lambda_1 + \mu_1 v'},$$

the values of v corresponding to $v'=0$, $v'=\infty$ are respectively

$$v = \lambda_2/\lambda_1 = \lambda, \qquad v = \mu_2/\mu_1 = \mu.$$

Since $-1/p$ and $-1/q$ are the two finite roots of $V'=0$, the other two roots of $V=0$ are

$$v=\frac{\lambda gp-\mu}{gp-1}\,,\ \text{corresponding to } v'=-\frac{1}{p}\,,$$

$$v=\frac{\lambda gq-\mu}{gq-1}\,,\ \text{corresponding to } v'=-\frac{1}{q}\,,$$

where

$$gp=\frac{(\mu-\lambda)^2}{\varkappa B}\sqrt{\frac{12g_2}{\theta}}\,,$$

$$gq=\frac{\varkappa(\mu-\lambda)^2}{B}\sqrt{\frac{12g_2}{\theta}}\,.$$

5. Definition of the s-Function.

Let the functional relation between z and w be defined by the differential equation

$$\left[\frac{dz}{dw}\right]^2=z(z+1)(\varkappa^{-2}z+1)\,,$$

and let sw be regarded as a solution of this equation, so that by definition

$$z=\mathrm{s}w\,,$$

$$\left[\frac{d\mathrm{s}w}{dw}\right]^2=\mathrm{s}w(\mathrm{s}w-1)(\varkappa^{-2}\mathrm{s}w-1)\,.$$

Then the differential equation

$$\frac{dv}{\sqrt{V}}=\left[\frac{\theta\varkappa^{-2}}{12g_2}\right]^{\frac{1}{4}}\frac{dz}{\sqrt{z(z+1)(\varkappa^{-2}z+1)}}$$

assumes the simpler form

$$\frac{dv}{\sqrt{V}}=\left[\frac{\theta\varkappa^{-2}}{12g_2}\right]^{\frac{1}{4}}dw\,,$$

and the rational relation between sw and v is

$$\mathrm{s}w=\frac{\varkappa(\mu-\lambda)^2}{B}\sqrt{\frac{12g_2}{\theta}}\cdot\frac{\lambda-v}{v-\mu}\,,$$

and symbolically

$$\int\frac{dv}{\sqrt{V}}=\left[\frac{\theta\varkappa^{-2}}{12g_2}\right]^{\frac{1}{4}}\mathrm{s}^{-1}\left[\frac{\varkappa(\mu-\lambda)^2}{B}\sqrt{\frac{12g_2}{\theta}}\cdot\frac{\lambda-v}{v-\mu}\right].$$

6. Definitions of the Functions sn_κ, cn_κ, dn_κ.

By the substitution $z = x^2$ the differential equation

$$\left[\frac{dz}{dw}\right]^2 = z\,(z+1)\,(\kappa^{-2}z+1)$$

becomes

$$4\left[\frac{dx}{dw}\right]^2 = (x^2+1)\,(\kappa^{-2}\,x^2+1)\,,$$

or, if $w = 2u$,

$$\left[\frac{dx}{du}\right]^2 = (x^2+1)\,(\kappa^{-2}\,x^2+1)\,;$$

and, by virtue of the previous linear transformation,

$$\frac{dv}{\sqrt{V}} = 2\left[\frac{\theta\kappa^{-2}}{12g_2}\right]^{\frac{1}{4}}\frac{dx}{\sqrt{(x^2+1)\,(\kappa^{-2}\,x^2+1)}} = 2\left[\frac{\theta\kappa^{-2}}{12g_2}\right]^{\frac{1}{4}}du\,.$$

We may now define outright

$$x = \mathrm{sn}_\kappa u\,,$$

or we may proceed as follows. Let $x = \sin_\kappa\varphi$; then

$$dx = \cos_\kappa\varphi\,d\varphi\,,$$

$$x^2+1 = \sin_\kappa{}^2\varphi+1\,,$$

$$\kappa^{-2}\,x^2+1 = \kappa^{-2}\sin^2\varphi+1 = \cos_\kappa{}^2\varphi\,,$$

and therefore

$$\frac{dx}{\sqrt{(x^2+1)\,(\kappa^{-2}\,x^2+1)}} = \frac{d\varphi}{\sqrt{\sin_\kappa{}^2\varphi+1}}\,.$$

If now

$$\frac{d\varphi}{\sqrt{\sin_\kappa{}^2\varphi+1}} = du\,,$$

φ is the amplitude of u with respect to the modulus κ ; symbolically

$$\varphi = \mathrm{am}_\kappa u\,.$$

With respect to the same modulus, let us also define

$$\sin_\kappa\varphi = \mathrm{sn}_\kappa u\,,$$

$$\cos_\kappa\varphi = \mathrm{cn}_\kappa u\,,$$

$$\sqrt{\sin_\kappa{}^2\varphi+1} = \mathrm{dn}_\kappa u\,;$$

it then follows that

$$\mathrm{cn}_\kappa{}^2 u - \kappa^{-2}\,\mathrm{sn}_\kappa{}^2 u = 1\,,$$

$$\mathrm{dn}_\kappa{}^2 u - \mathrm{sn}_\kappa{}^2 u = 1\,.$$

The differentials of these functions obviously are

$$d\mathrm{am}_\kappa u = \mathrm{dn}_\kappa u \, . \, du \, ,$$
$$d\mathrm{sn}_\kappa u = \mathrm{cn}_\kappa u \, . \, \mathrm{dn}_\kappa u \, . \, du \, ,$$
$$d\mathrm{cn}_\kappa u = \varkappa^{-2} \mathrm{sn}_\kappa u \, . \, \mathrm{dn}_\kappa u \, . \, du \, ,$$
$$d\mathrm{dn}_\kappa u = \mathrm{sn}_\kappa u \, . \, \mathrm{cn}_\kappa u \, . \, du \, .$$

7. Transition to Cyclo- and Hyperbo-Elliptic Forms.

Since $\sin_\kappa \varphi = i\varkappa \sin(\varphi / i\varkappa)$,

$$\therefore \quad \sqrt{\sin_\kappa^2 \varphi + 1} = \sqrt{1 - \varkappa^2 \sin_\kappa^2 (\varphi / i\varkappa)}$$

and
$$\frac{d\varphi}{\sqrt{\sin_\kappa^2 \varphi + 1}} = i\varkappa \frac{d(\varphi / i\varkappa)}{\sqrt{1 - \varkappa^2 \sin^2(\varphi / i\varkappa)}} = du \, ;$$

$$\therefore \quad \frac{\varphi}{i\varkappa} = \mathrm{am} \frac{u}{i\varkappa} \, ,$$

that is,
$$\mathrm{am}_\kappa u = i\varkappa \, \mathrm{am} \frac{u}{i\varkappa} \, .$$

Hence
$$\mathrm{sn}_\kappa u = i\varkappa \sin \frac{\varphi}{i\varkappa} = i\varkappa \, \mathrm{sn} \frac{u}{i\varkappa} \, ,$$

$$\mathrm{cn}_\kappa u = \cos \frac{\varphi}{i\varkappa} = \mathrm{cn} \frac{u}{i\varkappa} \, ,$$

and
$$\mathrm{dn}_\kappa u = \sqrt{1 - \varkappa^2 \sin^2(\varphi / i\varkappa)} = \mathrm{dn}_\kappa \frac{u}{i\varkappa} \, .$$

Similarly, since $\sin_\kappa \varphi = \varkappa \sinh(\varphi / \varkappa)$,

$$\therefore \quad \sqrt{\sin_\kappa^2 \varphi + 1} = \sqrt{\varkappa^2 \sinh^2(\varphi / \varkappa) + 1} \, ,$$

and
$$\frac{d\varphi}{\sqrt{\sin_\kappa^2 \varphi + 1}} = \varkappa \frac{d(\varphi / \varkappa)}{\sqrt{\varkappa^2 \sinh^2(\varphi / \varkappa) + 1}} = du \, .$$

Hence, if the hyperbolic forms, corresponding to am, sm, cn, dn, be denoted by hm, hs, hc, hd, respectively, we may define

$$\frac{\varphi}{\varkappa} = \mathrm{hm} \frac{u}{\varkappa} \, ,$$

that is
$$\mathrm{am}_\kappa u = \varkappa \, \mathrm{hm} \frac{u}{\varkappa} \, ;$$

also
$$\mathrm{sn}_\kappa u = \varkappa \sinh \frac{\varphi}{\varkappa} = \varkappa \, \mathrm{hs} \frac{u}{\varkappa} \, ,$$

$$\mathrm{cn}_\kappa u = \cosh \frac{\varphi}{\varkappa} = \mathrm{hc} \frac{u}{\varkappa} \, ,$$

and
$$\mathrm{dn}_\kappa u = \sqrt{\varkappa^2 \sinh^2(\varphi / \varkappa) + 1} = \mathrm{hd} \frac{u}{\varkappa} \, .$$

Hence also, writing for the moment $w = u/\varkappa$,

$$\operatorname{hm} w = i \operatorname{am} \frac{w}{i},$$

$$i \operatorname{hm} w = \operatorname{am} iw,$$

$$i \operatorname{hs} w = \operatorname{sn} iw,$$

$$\operatorname{hc} w = \operatorname{cn} iw,$$

$$\operatorname{hd} w = \operatorname{dn} iw.$$

I have given a detailed account of these hyperbolic forms in a paper on Hyperbo-Elliptic Functions, which appeared in the *Publications of the Astronomical Society of the Pacific*, Vol. I, pp. 177 *et seq.*

8. Further Development of the Theory.

We may now pass on to further details, and, *pari passu* with the successive steps of the theory as now known, produce the usual formulæ, modified in accordance with the fundamental principles here laid down. For example, we may show that

$$\operatorname{sn}_\kappa 0 = 0, \quad \operatorname{cn}_\kappa 0 = 1, \quad \operatorname{dn}_\kappa 0 = 1,$$

that

$$\operatorname{sn}_\kappa(-u) = -\operatorname{sn}_\kappa u, \quad \operatorname{cn}_\kappa(-u) = \operatorname{cn}_\kappa u, \quad \operatorname{dn}_\kappa(-u) = \operatorname{dn}_\kappa u;$$

that if

$$\int_0^{i\kappa} \frac{dx}{\sqrt{(x^2+1)(\kappa^{-2}x^2+1)}} = i\varkappa K,$$

$$\int_0^{i\kappa'} \frac{dx}{\sqrt{(x^2+1)(\kappa^{-2}x^2+1)}} = i\varkappa' K', \qquad [\varkappa'^2 = 1 - \varkappa^2]$$

then

$$\operatorname{sn}_\kappa i\varkappa K = i\varkappa. \quad \operatorname{cn}_\kappa i\varkappa K = 0, \quad \operatorname{dn}_\kappa i\varkappa K = \varkappa',$$

$$\operatorname{sn}_\kappa \varkappa(iK + K') = i, \quad \operatorname{cn}_\kappa \varkappa(iK + K') = \frac{i\varkappa'}{\varkappa}, \quad \operatorname{dn}_\kappa \varkappa(iK + K') = 0;$$

and that the addition equations for the functions s and sn_κ are

$$\frac{\operatorname{s}(u+v)}{\operatorname{s}u \,.\, \operatorname{s}v} = \left[\frac{\operatorname{s}u - \operatorname{s}v}{\operatorname{s}u\operatorname{s}'v - \operatorname{s}v\operatorname{s}'u}\right]^2,$$

$$\operatorname{sn}_\kappa(u+v) = \frac{\operatorname{sn}_\kappa^2 u - \operatorname{sn}_\kappa^2 v}{\operatorname{sn}_\kappa u \operatorname{sn}_\kappa' v - \operatorname{sn}_\kappa v \operatorname{sn}_\kappa' u}.$$

etc. etc.